Greedy algorithms

Shortest paths in weighted graphs

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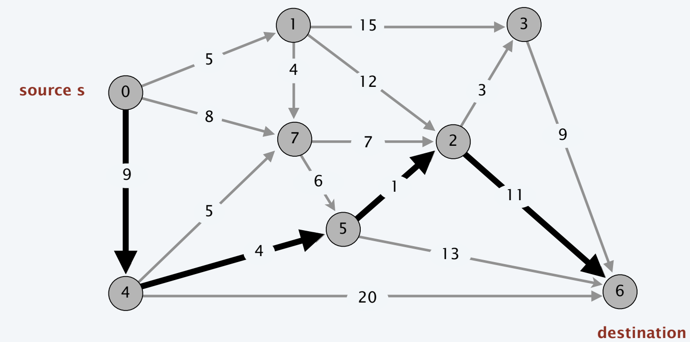
Some slides created by or adapted from Dr. Kevin Wayne. For more information see `http://www.cs.princeton.edu/~wayne/kleinberg-tardos`. Some code reused from Python Algorithms by Magnus Lie Hetland.

Shortest-paths problem

Problem. Given a digraph $G = (V, E)$, edge lengths $\ell_e \geq 0$, source $s \in V$, and destination $t \in V$, find the shortest directed path from s to t.



length of path = 9 + 4 + 1 + 11 = 25

Car navigation

Shortest path applications

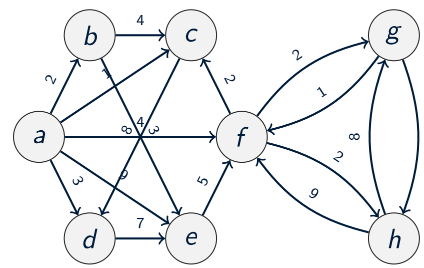
- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in LaTeX.
- Urban traffic planning.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

The many cases of finding shortest paths

- We've already seen how to calculate the shortest path in an unweighted graph (BFS traversal)
- We'll now study how to compute the shortest path in different circumstances for *weighted* graphs
  1. Single-source shortest path on a weighted DAG
  2. Single-source shortest path on a weighted graph with nonnegative weights (Dijkstra's algorithm)

Weighted Graph Data Structures



Nested Adjacency Dictionaries w/ Edge Weights

```
N = {
 'a':{'b':2,'c':1,'d':3,'e':9,'f':4},
 'b':{'c':4,'e':3},
 'c':{'d':8},
 'd':{'e':7},
 'e':{'f':5},
 'f':{'c':2,'g':2,'h':2},
 'g':{'f':1,'h':6},
 'h':{'f':9,'g':8}
}
>>> 'b' in N['a'] # Neighborhood membersh
True
>>> len(N['f'])   # Degree
3
>>> N['a']['b']
# Edge weight for (a, b)
2
```


Shortest paths in DAGs

- Recursive approach to finding the shortest path from a to z
  1. Assume we already know the distance $d(v)$ to z for each of a's neighbors $v \in G[a]$
  2. Select the neighbor v that minimizes $d(v) + W(a, v)$

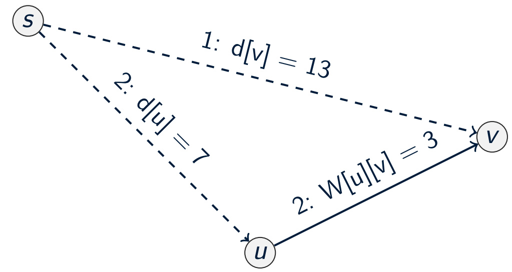
Recursive solution to finding shortest path in DAGs

```
def rec_dag_sp(W, s, t):      #Shortest path from s to t
    @memo                     #Memoize f
    def d(u):                 #Distance from u to t
        if u == t: return 0#We're there!
        # Return the best of every first step
        return min(W[u][v]+d(v) for v in W[u])
    return d(s)               #Apply f to actual start node
```

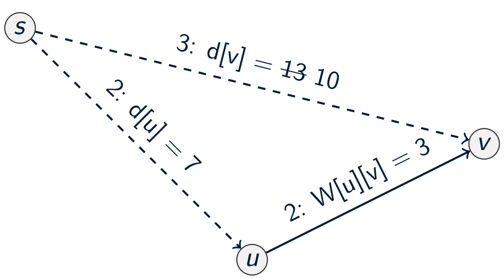
Shortest paths in DAGs: Iterative approach

- The iterative solution is a bit more complicated
  1. We must start with a topological sort
  2. Keep track of an upper bound on the distance from *a* to each node, initialized to ∞
  3. Go through each vertex and *relax* the distance estimate by inspecting the path from the vertex to its neighbor
- In general, *relaxing* an edge (u, v) consists of testing whether we can shorten the path to v found so far by going through u; if we can, we update $d[v]$ with the new value
- Running time: $\Theta(m + n)$

Relaxing edges



Relaxing edges

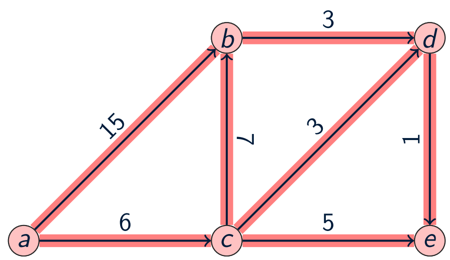



```
inf = float('inf')
def relax(W, u, v, D, P):
    d = D.get(u,inf) + W[u][v]# Possible shortcut estimate
    if d < D.get(v,inf):        # Is it really a shortcut?
        D[v], P[v] = d, u       # Update estimate and parent
        return True             # There was a change!
```


Iterative solution to finding shortest path in DAGs

```
def dag_sp(W, s, t):           #Shortest path from s to t
    d = {u:float('inf') for u in W}          # Distance esti
    d[s] = 0                   #Start node: Zero distance
    for u in topsort(W):       #In top-sorted order...
        if u == t: break       #Have we arrived?
        for v in W[u]:         #For each out-edge ...
            d[v] = min(d[v], d[u] + W[u][v])     # Relax the edg
    return d[t]                #Distance to t (from s)
```

Shortest-paths on weighted DAG example



Topological sort: a, c, b, d, e

Node	d[Node]: upper bd. dist. from *a*					
	init.	1 (u=a)	2 (u=c)	3 (u=b)	4 (u=d)	5 (u=e)
a	0	0	0	0	0	0
b	∞	15	13	13	13	13
c	∞	6	6	6	6	6
d	∞	∞	9	9	9	9
e	∞	∞	11	11	10	10

Shortest-paths on weighted DAG: exercise

But what if there are cycles?

- With a DAG, we can select the order in which to visit nodes based on the topological sort
- With cycles we can't easily determine the best order
- If there are no negative edges, we can traverse from the starting vertex, visiting nodes in order of their estimated distance from the starting vertex
- In Dijkstra's algorithm, we use a priority queue based on minimum estimated distance from the source to select which vertices to visit
- Running time: $\Theta((m+n) \lg n)$
- Dijkstra's algorithm combines approaches seen in other algorithms
  1. Node discovery: bit like breadth-first traversal
  2. Node visitation: selected using priority queue
  3. Shortest path calculation: uses relaxation as in algorithm for shortest paths in DAGs

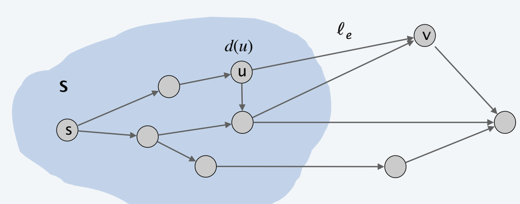
Dijkstra's algorithm

Greedy approach. Maintain a set of explored nodes S for which algorithm has determined the shortest path distance $d(u)$ from s to u.

- Initialize $S = \{ s \}$, $d(s) = 0$.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some node u in explored part, followed by a single edge (u, v)



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Dijkstra's algorithm

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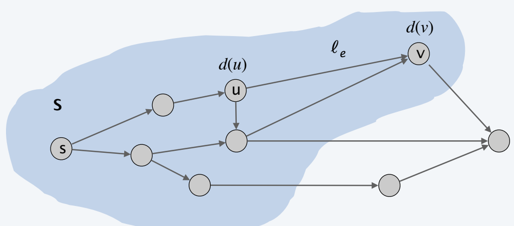
$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

add v to S, and set $d(v) = \pi(v)$.

shortest path to some node u in explored part, followed by a single edge (u, v)



Dijkstra's algorithm: proof of correctness

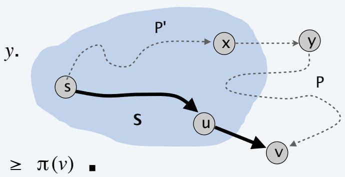
Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest $s \leadsto u$ path.

Pf. [by induction on $|S|$]

Base case: $|S| = 1$ is easy since $S = \{ s \}$ and $d(s) = 0$.

Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S, and let (u, v) be the final edge.
- The shortest $s \leadsto u$ path plus (u, v) is an $s \leadsto v$ path of length $\pi(v)$.
- Consider any $s \leadsto v$ path P. We show that it is no shorter than $\pi(v)$.
- Let (x, y) be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it reaches y.



$$\ell(P) \geq \ell(P') + \ell(x, y) \geq d(x) + \ell(x, y) \geq \pi(y) \geq \pi(v) \ ▪$$

↑ nonnegative lengths ↑ inductive hypothesis ↑ definition of $\pi(y)$ ↑ Dijkstra chose v instead of y

Dijkstra's algorithm: efficient implementation

Critical optimization 1. For each unexplored node v, explicitly maintain $\pi(v)$ instead of computing directly from formula:

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e .$$

- For each $v \notin S$, $\pi(v)$ can only decrease (because S only increases).
- More specifically, suppose u is added to S and there is an edge (u, v) leaving u. Then, it suffices to update:

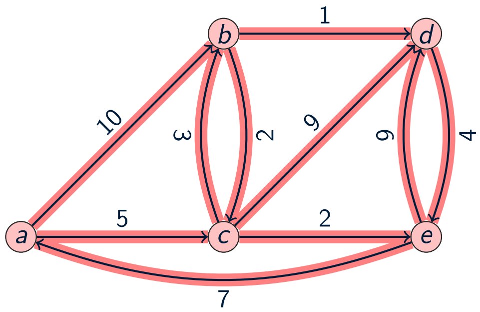
$$\pi(v) = \min \{ \pi(v),\ d(u) + \ell(u, v) \}$$

Critical optimization 2. Use a priority queue to choose the unexplored node that minimizes $\pi(v)$.

Dijkstra's algorithm

```
from heapq import heappush, heappop
def dijkstra(G, s):
    D, P, Q, S = {s:0}, {}, [(0,s)], set()     # Est., tree, q
    while Q:                                   # Still unprocessed nodes?
        _, u = heappop(Q)                      # Node with lowest estimate
        if u in S: continue                    # Already visited? Skip it
        S.add(u)                               # We've visited it now
        for v in G[u]:                         # Go through all its neighbors
            relax(G, u, v, D, P)               # Relax the out-edge
            heappush(Q, (D[v], v))# Add to queue, w/est. as pri
    return D, P                                # Final D and P returned
```

Dijkstra's algorithm example



Node	init.	1 (u=a)	2 (u=c)	3 (u=e)	4 (u=b)	5 (u=d)
	`d[Node]`: upper bd. dist. from a					
a	0	0	0	0	0	0
b	∞	10	8	8	8	8
c	∞	5	5	5	5	5
d	∞	∞	14	13	9	9
e	∞	∞	7	7	7	7

Dijkstra's algorithm: exercise