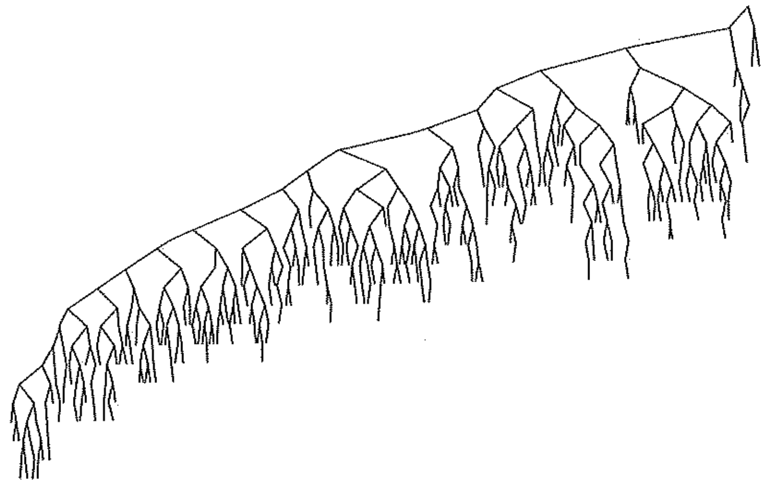


Figure 4.27 Binary search tree after $\Theta(N^2)$ `insert/remove` pairs

We could try to eliminate the problem by randomly choosing between the smallest element in the right subtree and the largest in the left when replacing the deleted element. This apparently eliminates the bias and should keep the trees balanced, but nobody has actually proved this. In any event, this phenomenon appears to be mostly a theoretical novelty, because the effect does not show up at all for small trees, and stranger still, if $o(N^2)$ `insert/remove` pairs are used, then the tree seems to gain balance!

The main point of this discussion is that deciding what "average" means is generally extremely difficult and can require assumptions that may or may not be valid. In the absence of deletions, or when lazy deletion is used, we can conclude that the average running times of the operations above are $O(\log N)$. Except for strange cases like the one discussed above, this result is very consistent with observed behavior.

If the input comes into a tree presorted, then a series of `inserts` will take quadratic time and give a very expensive implementation of a linked list, since the tree will consist only of nodes with no left children. One solution to the problem is to insist on an extra structural condition called *balance:* no node is allowed to get too deep.

There are quite a few general algorithms to implement balanced trees. Most are quite a bit more complicated than a standard binary search tree, and all take longer on average for updates. They do, however, provide protection against the embarrassingly simple cases. Below, we will sketch one of the oldest forms of balanced search trees, the AVL tree.

A second, newer method is to forgo the balance condition and allow the tree to be arbitrarily deep, but after every operation, a restructuring rule is applied that tends to make future operations efficient. These types of data structures are generally classified as **self-adjusting**. In the case of a binary search tree, we can no longer guarantee an $O(\log N)$ bound on any single operation, but can show that any *sequence* of M operations takes total time $O(M \log N)$ in the worst case. This is generally sufficient protection against a bad worst case. The data structure we will discuss is known as a *splay tree;* its analysis is fairly intricate and is discussed in Chapter 11.

4.4 AVL Trees

An AVL (Adelson-Velskii and Landis) tree is a binary search tree with a **balance condition**. The balance condition must be easy to maintain, and it ensures that the depth of the tree is $O(\log N)$. The simplest idea is to require that the left and right subtrees have the same height. As Figure 4.28 shows, this idea does not force the tree to be shallow.

Another balance condition would insist that every node must have left and right subtrees of the same height. If the height of an empty subtree is defined to be -1 (as is usual), then only perfectly balanced trees of $2^h - 1$ nodes would satisfy this criterion. Thus, although this guarantees trees of small depth, the balance condition is too rigid to be useful and needs to be relaxed.

An AVL **tree** is identical to a binary search tree, except that for every node in the tree, the height of the left and right subtrees can differ by at most 1. (The height of an empty tree is defined to be -1.) In Figure 4.29 the tree on the left is an AVL tree, but the tree on the right is not. Height information is kept for each node (in the node structure). It can be shown that the height of an AVL tree is at most roughly $1.44 \log(N + 2) - 1.328$, but in practice it is only slightly more than $\log N$. As an example, the AVL tree of height 9 with

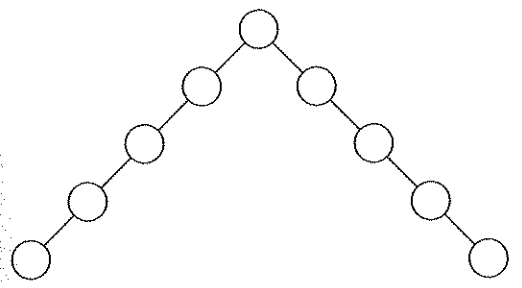


Figure 4.28 A bad binary tree. Requiring balance at the root is not enough.

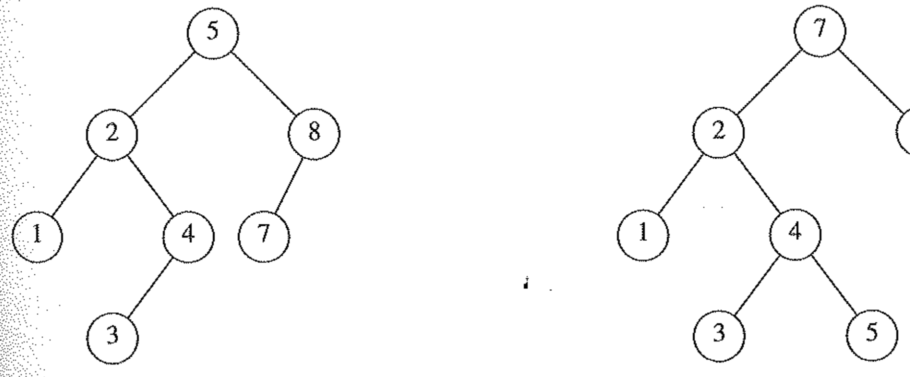


Figure 4.29 Two binary search trees. Only the left tree is AVL.

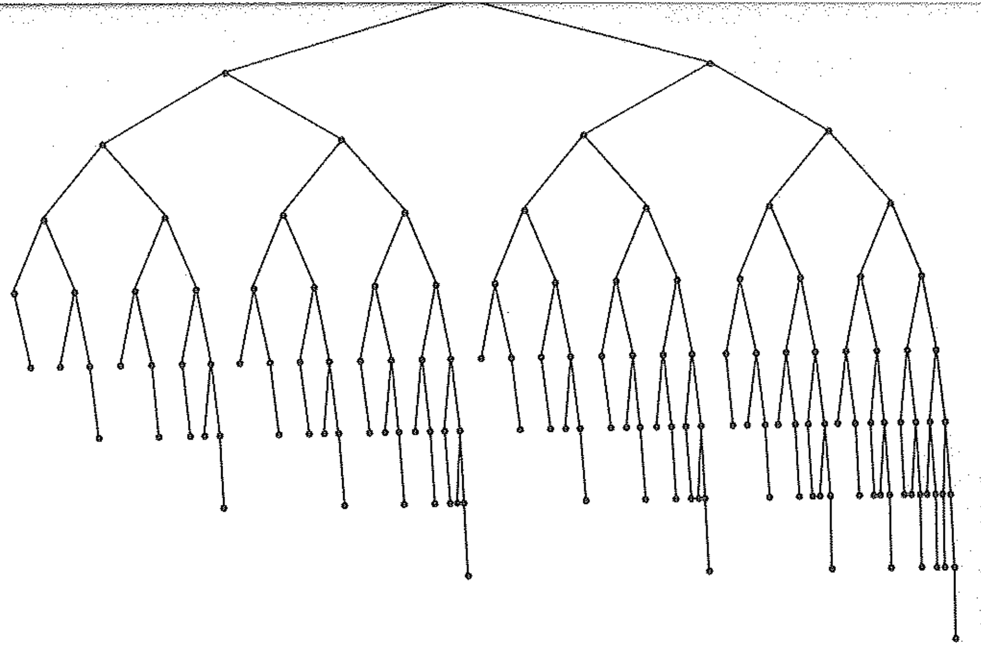


Figure 4.30 Smallest AVL tree of height 9

the fewest nodes (143) is shown in Figure 4.30. This tree has as a left subtree an AVL tree of height 7 of minimum size. The right subtree is an AVL tree of height 8 of minimum size. This tells us that the minimum number of nodes, $S(h)$, in an AVL tree of height h is given by $S(h) = S(h-1) + S(h-2) + 1$. For $h = 0$, $S(h) = 1$. For $h = 1$, $S(h) = 2$. The function $S(h)$ is closely related to the Fibonacci numbers, from which the bound claimed above on the height of an AVL tree follows.

Thus, all the tree operations can be performed in $O(\log N)$ time, except possibly insertion (we will assume lazy deletion). When we do an insertion, we need to update all the balancing information for the nodes on the path back to the root, but the reason that insertion is potentially difficult is that inserting a node could violate the AVL tree property. (For instance, inserting 6 into the AVL tree in Figure 4.29 would destroy the balance condition at the node with key 8.) If this is the case, then the property has to be restored before the insertion step is considered over. It turns out that this can always be done with a simple modification to the tree, known as a **rotation**.

After an insertion, only nodes that are on the path from the insertion point to the root might have their balance altered because only those nodes have their subtrees altered. As we follow the path up to the root and update the balancing information, we may find a node whose new balance violates the AVL condition. We will show how to rebalance the tree at the first (i.e., deepest) such node, and we will prove that this rebalancing guarantees that the entire tree satisfies the AVL property.

and a height imbalance requires that α's two subtrees' height differ by two, it is easy to see that a violation might occur in four cases:

1. An insertion into the left subtree of the left child of α.
2. An insertion into the right subtree of the left child of α.
3. An insertion into the left subtree of the right child of α.
4. An insertion into the right subtree of the right child of α.

Cases 1 and 4 are mirror image symmetries with respect to α, as are cases 2 and 3. Consequently, as a matter of theory, there are two basic cases. From a programming perspective, of course, there are still four cases.

The first case, in which the insertion occurs on the "outside" (i.e., left–left or right–right), is fixed by a **single rotation** of the tree. The second case, in which the insertion occurs on the "inside" (i.e., left–right or right–left) is handled by the slightly more complex **double rotation**. These are fundamental operations on the tree that we'll see used several times in balanced-tree algorithms. The remainder of this section describes these rotations, proves that they suffice to maintain balance, and gives a casual implementation of the AVL tree. Chapter 12 describes other balanced-tree methods with an eye toward a more careful implementation.

4.4.1 Single Rotation

Figure 4.31 shows the *single rotation* that fixes case 1. The before picture is on the left, and the after is on the right. Let us analyze carefully what is going on. Node k_2 violates the AVL balance property because its left subtree is two levels deeper than its right subtree (the dashed lines in the middle of the diagram mark the levels). The situation depicted is the only possible case 1 scenario that allows k_2 to satisfy the AVL property before an insertion but violate it afterwards. Subtree X has grown to an extra level, causing it to be exactly two levels deeper than Z. Y cannot be at the same level as the new X because then k_2 would have been out of balance *before* the insertion, and Y cannot be at the same level as Z because then k_1 would be the first node on the path toward the root that was in violation of the AVL balancing condition.

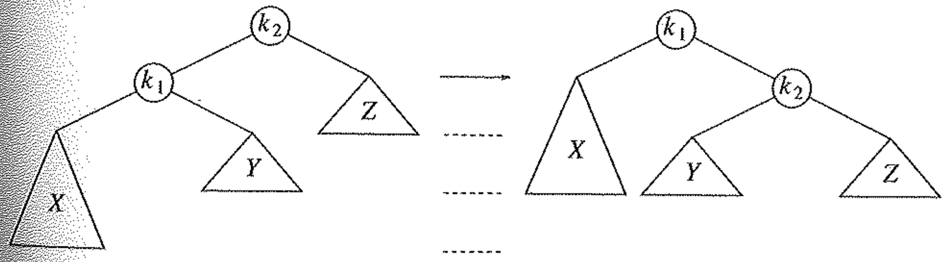


Figure 4.31 Single rotation to fix case 1

To ideally rebalance the tree, we would like to move X up a level and Z down a level. Note that this is actually more than the AVL property would require. To do this, we rearrange nodes into an equivalent tree as shown in the second part of Figure 4.31. Here is an abstract scenario: visualize the tree as being flexible, grab the child node k_1, close your eyes, and shake it, letting gravity take hold. The result is that k_1 will be the new root. The binary search tree property tells us that in the original tree $k_2 > k_1$, so k_2 becomes the right child of k_1 in the new tree. X and Z remain as the left child of k_1 and right child of k_2, respectively. Subtree Y, which holds items that are between k_1 and k_2 in the original tree, can be placed as k_2's left child in the new tree and satisfy all the ordering requirements.

As a result of this work, which requires only a few link changes, we have another binary search tree that is an AVL tree. This happens because X moves up one level, Y stays at the same level, and Z moves down one level. k_2 and k_1 not only satisfy the AVL requirements, but they also have subtrees that are exactly the same height. Furthermore, the new height of the entire subtree is *exactly the same* as the height of the original subtree prior to the insertion that caused X to grow. Thus no further updating of heights on the path to the root is needed, and consequently *no further rotations are needed.* Figure 4.32 shows that after the insertion of 6 into the original AVL tree on the left, node 8 becomes unbalanced. Thus, we do a single rotation between 7 and 8, obtaining the tree on the right.

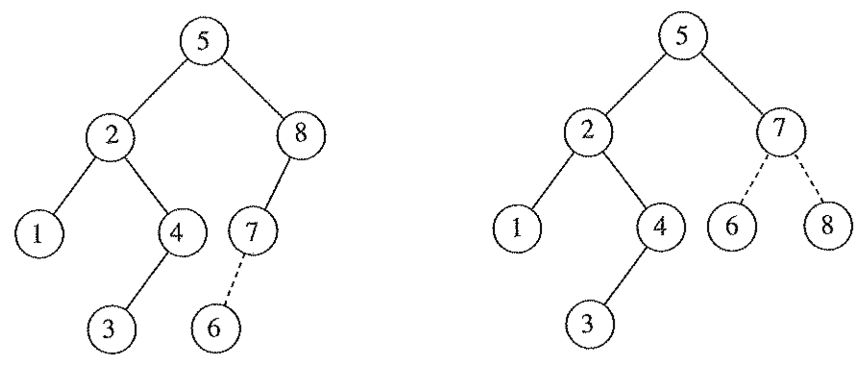


Figure 4.32 AVL property destroyed by insertion of 6, then fixed by a single rotation

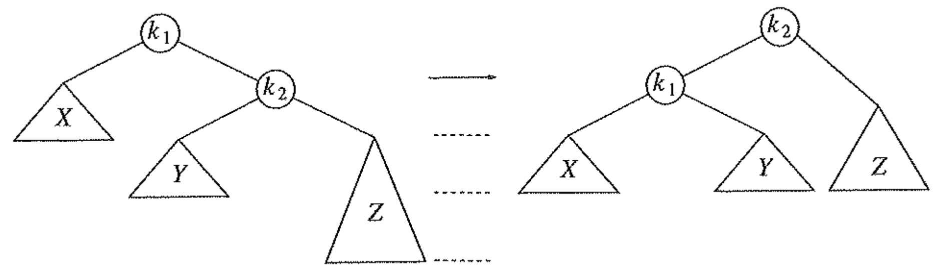
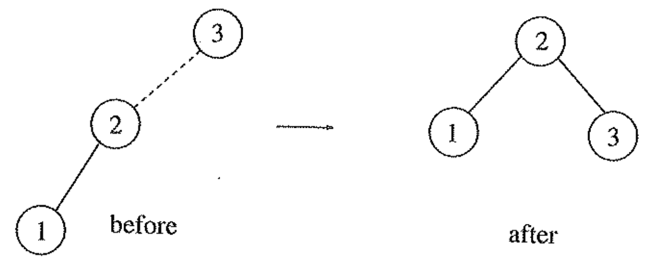
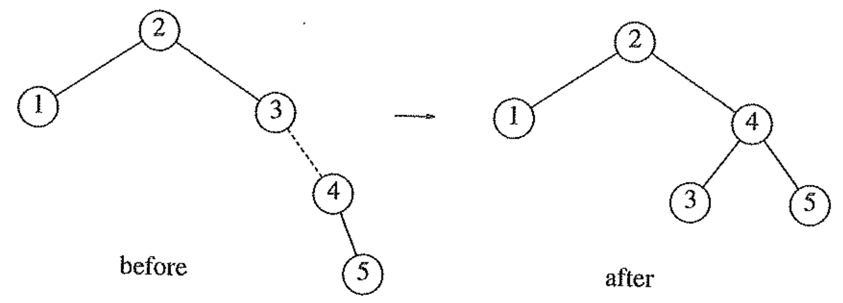


Figure 4.33 Single rotation fixes case 4

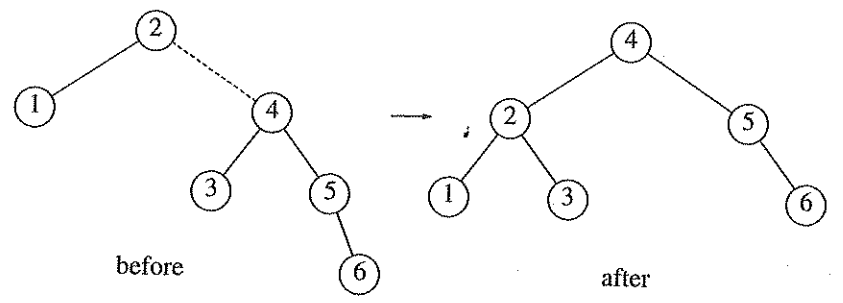
As we mentioned earlier, case 4 represents a symmetric case. Figure 4.33 shows how a single rotation is applied. Let us work through a rather long example. Suppose we start with an initially empty AVL tree and insert the items 3, 2, 1, and then 4 through 7 in sequential order. The first problem occurs when it is time to insert 1 because the AVL property is violated at the root. We perform a single rotation between the root and its left child to fix the problem. Here are the before and after trees:



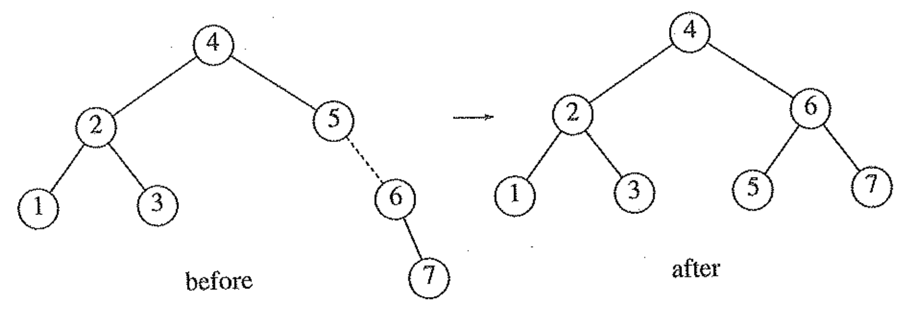
A dashed line joins the two nodes that are the subject of the rotation. Next we insert 4, which causes no problems, but the insertion of 5 creates a violation at node 3 that is fixed by a single rotation. Besides the local change caused by the rotation, the programmer must remember that the rest of the tree has to be informed of this change. Here this means that 2's right child must be reset to link to 4 instead of 3. Forgetting to do so is easy and would destroy the tree (4 would be inaccessible).



Next we insert 6. This causes a balance problem at the root, since its left subtree is of height 0 and its right subtree would be height 2. Therefore, we perform a single rotation at the root between 2 and 4.



The rotation is performed by making 2 a child of 4 and 4's original left subtree the new right subtree of 2. Every item in this subtree must lie between 2 and 4, so this transformation makes sense. The next item we insert is 7, which causes another rotation:



4.4.2 Double Rotation

The algorithm described above has one problem: as Figure 4.34 shows, it does not work for cases 2 or 3. The problem is that subtree Y is too deep, and a single rotation does not make it any less deep. The *double rotation* that solves the problem is shown in Figure 4.35.

The fact that subtree Y in Figure 4.34 has had an item inserted into it guarantees that it is nonempty. Thus, we may assume that it has a root and two subtrees. Consequently, the tree may be viewed as four subtrees connected by three nodes. As the diagram suggests,

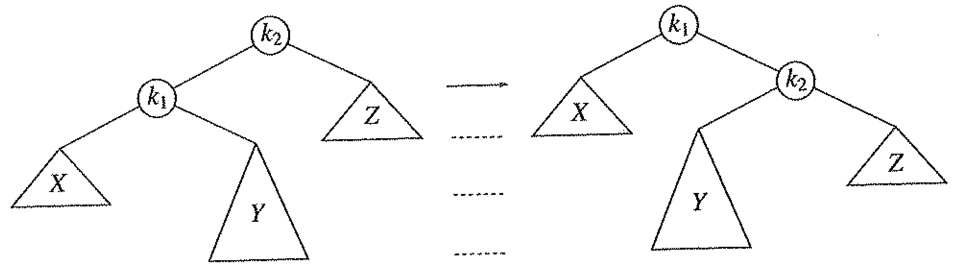


Figure 4.34 Single rotation fails to fix case 2

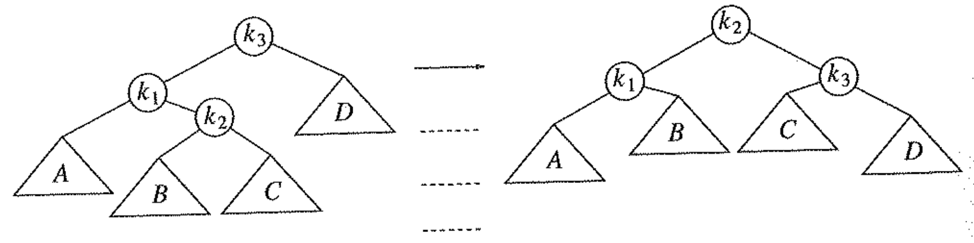


Figure 4.35 Left–right double rotation to fix case 2

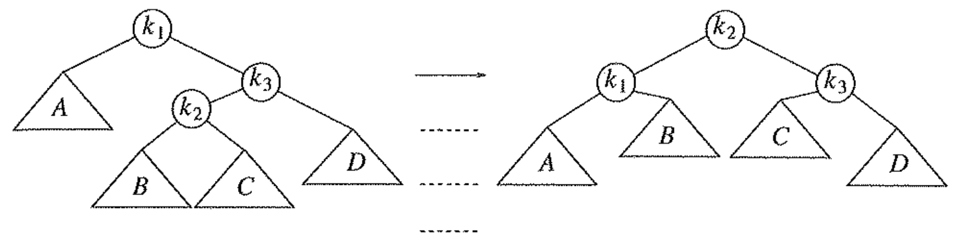
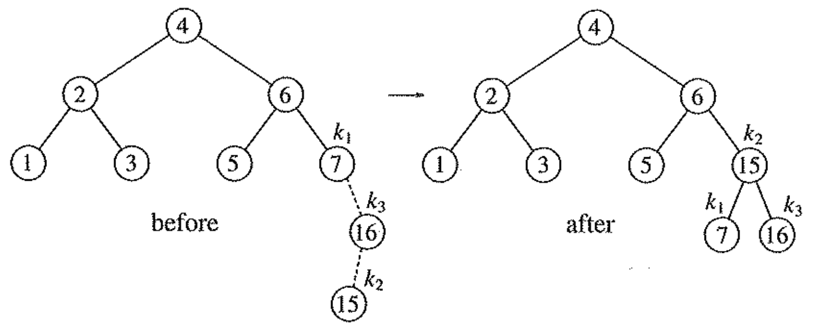


Figure 4.36 Right–left double rotation to fix case 3

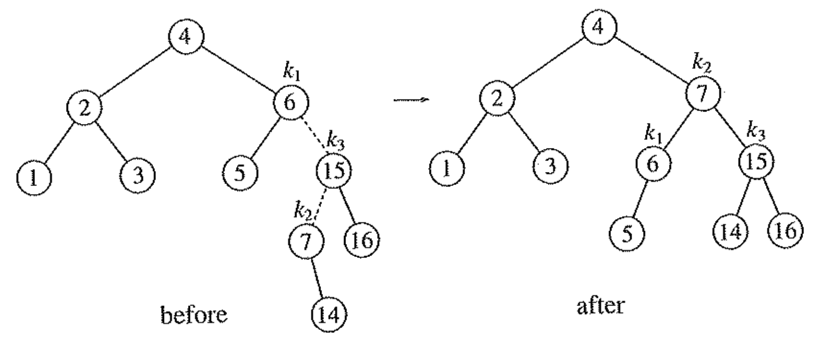
exactly one of tree B or C is two levels deeper than D (unless all are empty), but we cannot be sure which one. It turns out not to matter; in Figure 4.35, both B and C are drawn at $1\frac{1}{2}$ levels below D.

To rebalance, we see that we cannot leave k_3 as the root, and a rotation between k_3 and k_1 was shown in Figure 4.34 to not work, so the only alternative is to place k_2 as the new root. This forces k_1 to be k_2's left child and k_3 to be its right child, and it also completely determines the resulting locations of the four subtrees. It is easy to see that the resulting tree satisfies the AVL tree property, and as was the case with the single rotation, it restores the height to what it was before the insertion, thus guaranteeing that all rebalancing and height updating is complete. Figure 4.36 shows that the symmetric case 3 can also be fixed by a double rotation. In both cases the effect is the same as rotating between α's child and grandchild, and then between α and its new child.

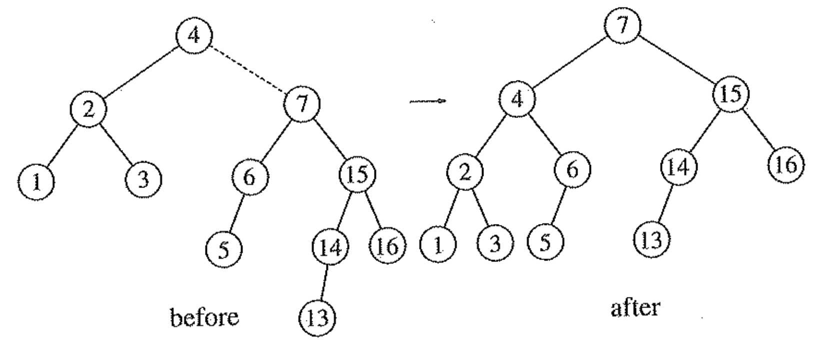
We will continue our previous example by inserting 10 through 16 in reverse order, followed by 8 and then 9. Inserting 16 is easy, since it does not destroy the balance property, but inserting 15 causes a height imbalance at node 7. This is case 3, which is solved by a right–left double rotation. In our example, the right–left double rotation will involve 7, 16, and 15. In this case, k_1 is the node with item 7, k_3 is the node with item 16, and k_2 is the node with item 15. Subtrees A, B, C, and D are empty.



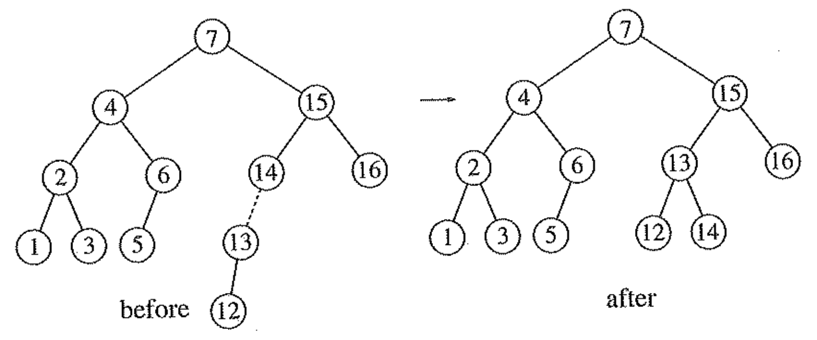
Next we insert 14, which also requires a double rotation. Here the double rotation that will restore the tree is again a right–left double rotation that will involve 6, 15, and 7. In this case, k_1 is the node with item 6, k_2 is the node with item 7, and k_3 is the node with item 15. Subtree A is the tree rooted at the node with item 5; subtree B is the empty subtree that was originally the left child of the node with item 7, subtree C is the tree rooted at the node with item 14, and finally, subtree D is the tree rooted at the node with item 16.



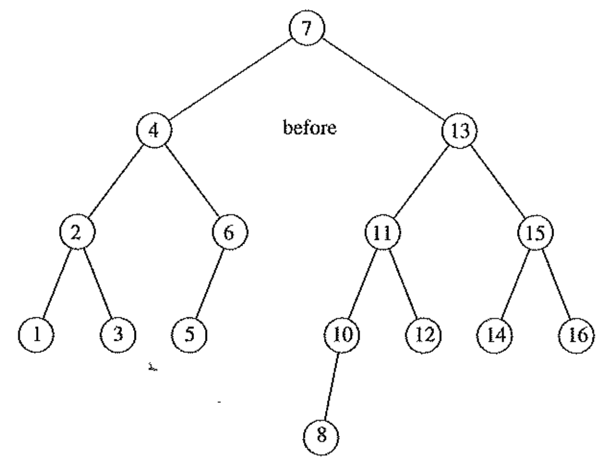
If 13 is now inserted, there is an imbalance at the root. Since 13 is not between 4 and 7, we know that the single rotation will work.



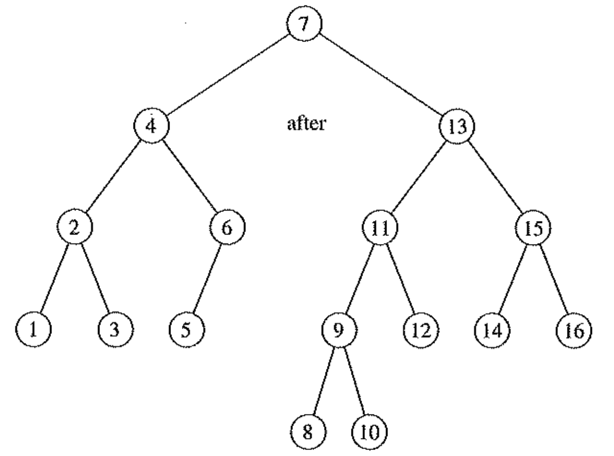
Insertion of 12 will also require a single rotation:



To insert 11, a single rotation needs to be performed, and the same is true for the subsequent insertion of 10. We insert 8 without a rotation creating an almost perfectly balanced tree:



Finally, we will insert 9 to show the symmetric case of the double rotation. Notice that 9 causes the node containing 10 to become unbalanced. Since 9 is between 10 and 8 (which is 10's child on the path to 9), a double rotation needs to be performed, yielding the following tree:



Let us summarize what happens. The programming details are fairly straightforward except that there are several cases. To insert a new node with item X into an AVL tree T, we recursively insert X into the appropriate subtree of T (let us call this T_{LR}). If the height of T_{LR} does not change, then we are done. Otherwise, if a height imbalance appears in T, we do the appropriate single or double rotation depending on X and the items in T and T_{LR}, update the heights (making the connection from the rest of the tree above), and are done. Since one rotation always suffices, a carefully coded nonrecursive version generally turns out to be significantly faster than the recursive version. However, nonrecursive versions are quite difficult to code correctly, so many programmers implement AVL trees recursively.

Another efficiency issue concerns storage of the height information. Since all that is really required is the difference in height, which is guaranteed to be small, we could get by with two bits (to represent $+1$, 0, -1) if we really try. Doing so will avoid repetitive calculation of balance factors but results in some loss of clarity. The resulting code is somewhat more complicated than if the height were stored at each node. If a recursive routine is written, then speed is probably not the main consideration. In this case, the slight speed advantage obtained by storing balance factors hardly seems worth the loss of clarity and relative simplicity. Furthermore, since most machines will align this to at least an 8-bit boundary anyway, there is not likely to be any difference in the amount of space used. An eight-bit `byte` will allow us to store absolute heights of up to 127. Since the tree is balanced, it is inconceivable that this would be insufficient (see the exercises).

With all this, we are ready to write the AVL routines. We show some of the code here; the rest is online. First, we need the `AvlNode` class. This is given in Figure 4.37. We also need a quick method to return the height of a node. This method is necessary to handle the annoying case of a `null` reference. This is shown in Figure 4.38. The basic insertion routine is easy to write, since it consists mostly of method calls (see Figure 4.39).

For the trees in Figure 4.40, `rotateWithLeftChild` converts the tree on the left to the tree on the right, returning a reference to the new root. `rotateWithRightChild` is symmetric. The code is shown in Figure 4.41.

The last method we will write will perform the double rotation pictured in Figure 4.42, for which the code is shown in Figure 4.43.

```
private static class AvlNode<AnyType>
{
        // Constructors
    AvlNode( AnyType theElement )
      { this( theElement, null, null ); }

    AvlNode( AnyType theElement, AvlNode<AnyType> lt, AvlNode<AnyType> rt )
      { element  = theElement; left = lt; right = rt; height   = 0; }

    AnyType           element;      // The data in the node
    AvlNode<AnyType>  left;         // Left child
    AvlNode<AnyType>  right;        // Right child
    int               height;       // Height
}
```

Figure 4.37 Node declaration for AVL trees

```
/**
 * Return the height of node t, or -1, if null.
 */
private int height( AvlNode<AnyType> t )
{
    return t == null ? -1 : t.height;
}
```

Figure 4.38 Method to compute height of an AVL node

```
/**
 * Internal method to insert into a subtree.
 * @param x the item to insert.
 * @param t the node that roots the subtree.
 * @return the new root of the subtree.
 */
private AvlNode<AnyType> insert( AnyType x, AvlNode<AnyType> t )
{
    if( t == null )
        return new AvlNode<AnyType>( x, null, null );

    int compareResult = compare( x, t.element );

    if( compareResult < 0 )
    {
        t.left = insert( x, t.left );
        if( height( t.left ) - height( t.right ) == 2 )
            if( compare( x, t.left.element ) < 0 )
                t = rotateWithLeftChild( t );
            else
                t = doubleWithLeftChild( t );
    }
    else if( compareResult > 0 )
    {
        t.right = insert( x, t.right );
        if( height( t.right ) - height( t.left ) == 2 )
            if( compare( x, t.right.element ) > 0 )
                t = rotateWithRightChild( t );
            else
                t = doubleWithRightChild( t );
    }
    else
        ;  // Duplicate; do nothing
    t.height = Math.max( height( t.left ), height( t.right ) ) + 1;
    return t;
}
```

Figure 4.39 Insertion into an AVL tree

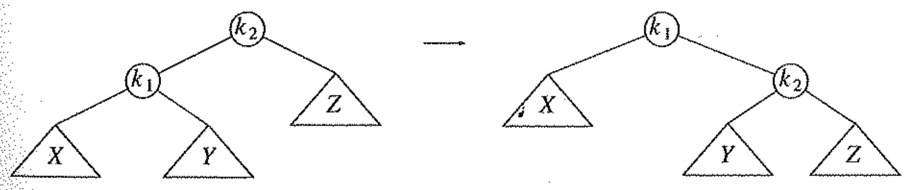


Figure 4.40 Single rotation

```
    /**
     * Rotate binary tree node with left child.
     * For AVL trees, this is a single rotation for case 1.
     * Update heights, then return new root.
     */
    private AvlNode<AnyType> rotateWithLeftChild( AvlNode<AnyType> k2 )
    {
        AvlNode<AnyType> k1 = k2.left;
        k2.left = k1.right;
        k1.right = k2;
        k2.height = Math.max( height( k2.left ), height( k2.right ) ) + 1;
        k1.height = Math.max( height( k1.left ), k2.height ) + 1;
        return k1;
    }
```

Figure 4.41 Routine to perform single rotation

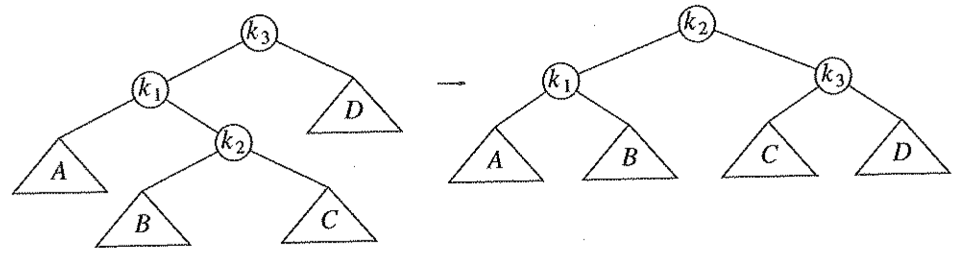


Figure 4.42 Double rotation

```
    /**
     * Double rotate binary tree node: first left child
     * with its right child; then node k3 with new left child.
     * For AVL trees, this is a double rotation for case 2.
     * Update heights, then return new root.
     */
    private AvlNode<AnyType> doubleWithLeftChild( AvlNode<AnyType> k3 )
    {
        k3.left = rotateWithRightChild( k3.left );
        return rotateWithLeftChild( k3 );
    }
```

Figure 4.43 Routine to perform double rotation

Deletion in AVL trees is somewhat more complicated than insertion, and is left as an exercise. Lazy deletion is probably the best strategy if deletions are relatively infrequent.

4.5 Splay Trees

We now describe a relatively simple data structure, known as a **splay tree**, that guarantees that any M consecutive tree operations starting from an empty tree take at most $O(M \log N)$ time. Although this guarantee does not preclude the possibility that any *single* operation might take $O(N)$ time, and thus the bound is not as strong as an $O(\log N)$ worst-case bound per operation, the net effect is the same: There are no bad input sequences. Generally, when a sequence of M operations has total worst-case running time of $O(Mf(N))$, we say that the **amortized** running time is $O(f(N))$. Thus, a splay tree has an $O(\log N)$ amortized cost per operation. Over a long sequence of operations, some may take more, some less.

Splay trees are based on the fact that the $O(N)$ worst-case time per operation for binary search trees is not bad, as long as it occurs relatively infrequently. Any one access, even if it takes $O(N)$, is still likely to be extremely fast. The problem with binary search trees is that it is possible, and not uncommon, for a whole sequence of bad accesses to take place. The cumulative running time then becomes noticeable. A search tree data structure with $O(N)$ worst-case time, but a *guarantee* of at most $O(M \log N)$ for any M consecutive operations, is certainly satisfactory, because there are no bad sequences.

If any particular operation is allowed to have an $O(N)$ worst-case time bound, and we still want an $O(\log N)$ amortized time bound, then it is clear that whenever a node is accessed, it must be moved. Otherwise, once we find a deep node, we could keep performing accesses on it. If the node does not change location, and each access costs $O(N)$, then a sequence of M accesses will cost $O(M \cdot N)$.

The basic idea of the splay tree is that after a node is accessed, it is pushed to the root by a series of AVL tree rotations. Notice that if a node is deep, there are many nodes on the path that are also relatively deep, and by restructuring we can make future accesses cheaper on all these nodes. Thus, if the node is unduly deep, then we want this restructuring to have the side effect of balancing the tree (to some extent). Besides giving a good time bound in theory, this method is likely to have practical utility, because in many applications, when a node is accessed, it is likely to be accessed again in the near future. Studies have shown that this happens much more often than one would expect. Splay trees also do not require the maintenance of height or balance information, thus saving space and simplifying the code to some extent (especially when careful implementations are written).

4.5.1 A Simple Idea (That Does Not Work)

One way of performing the restructuring described above is to perform single rotations, bottom up. This means that we rotate every node on the access path with its parent. As an example, consider what happens after an access (a `find`) on k_1 in the following tree.